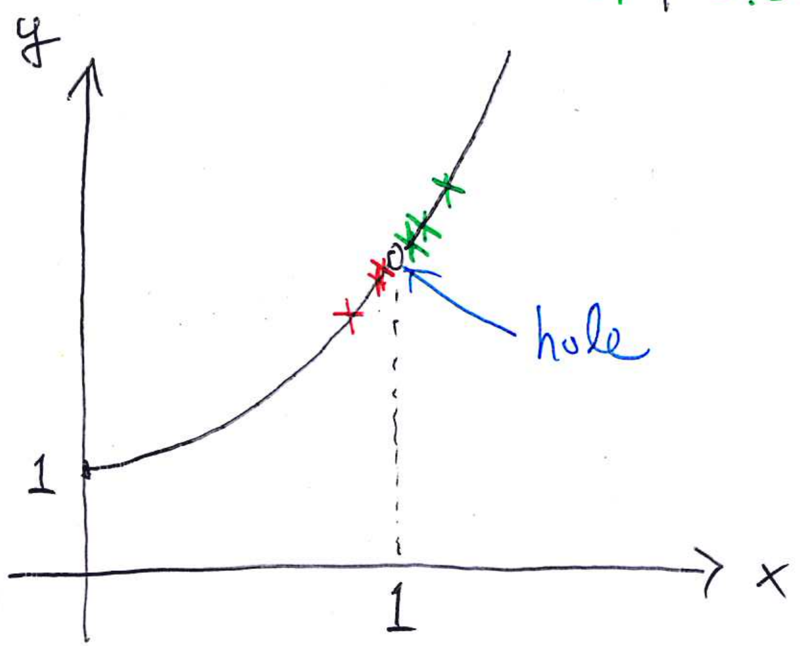
Week 3 Limit of Functions

eg $f(x) = \dfrac{x^3-1}{x-1}$

Domain : $\boxed{\mathbb{R} \setminus \{1\}}$

x	$f(x)$
0.9	2.71
0.99	2.9701
0.999	2.997001
1.001	3.003001
1.01	3.0301
1.1	3.31

Q What happens when x is very close to 1?



Observation

① As $x \to 1$ from the left (i.e $x < 1$)

$f(x) \to 3$

$$\underbrace{\lim_{x \to 1^-} f(x)}_{\text{Left-hand limit}} = 3$$

② Similarly, as $x \to 1$ from the right (i.e $x > 1$)

$f(x) \to 3$

$$\underbrace{\lim_{x \to 1^+} f(x)}_{\text{right-hand limit}} = 3$$

We can say that $\lim_{x \to 1} f(x) = 3$

Intuitive definition of limit of functions

Let $a, L \in \mathbb{R}$, $f(x)$ be a function. We say that

$$\begin{cases} \lim_{x\to a^-} f(x) = L \\ \lim_{x\to a^+} f(x) = L \\ \lim_{x\to a} f(x) = L \end{cases}$$ if $f(x)$ is close enough to L

when x is close enough to a and $\begin{cases} x < a \\ x > a \\ x \neq a \end{cases}$

Rmk

- Whether f is defined at a or the value of f at a is not important for $\lim_{x\to a^\pm} f(x)$ or $\lim_{x\to a} f(x)$.
- $$\boxed{\begin{aligned} &\lim_{x\to a} f(x) = L \\ \iff &\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L \end{aligned}}$$

ε-δ definition of limit of functions

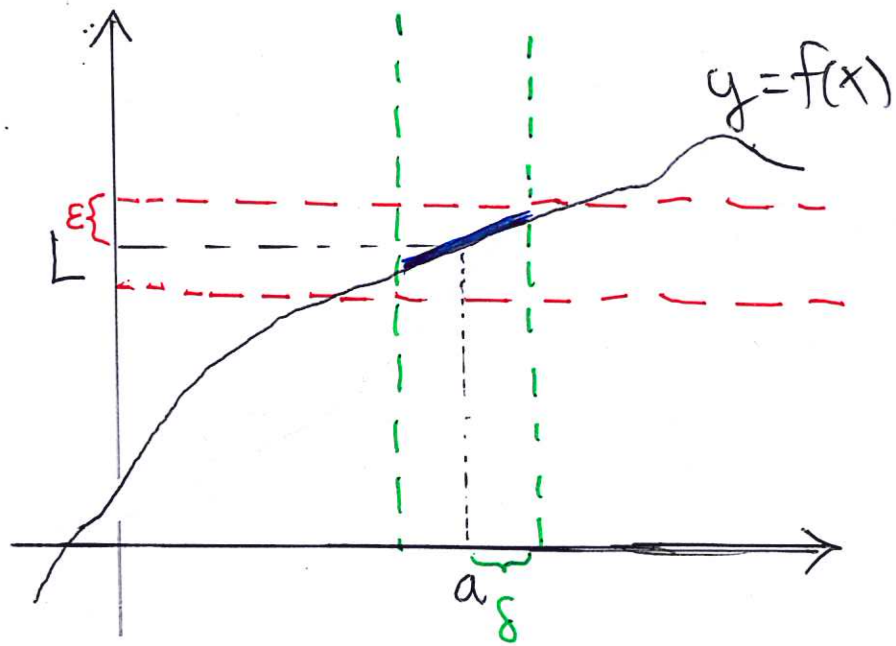
(Optional)

Def $\lim_{x\to a} f(x) = L$

$\iff \forall \varepsilon > 0, \exists \delta > 0$ such that

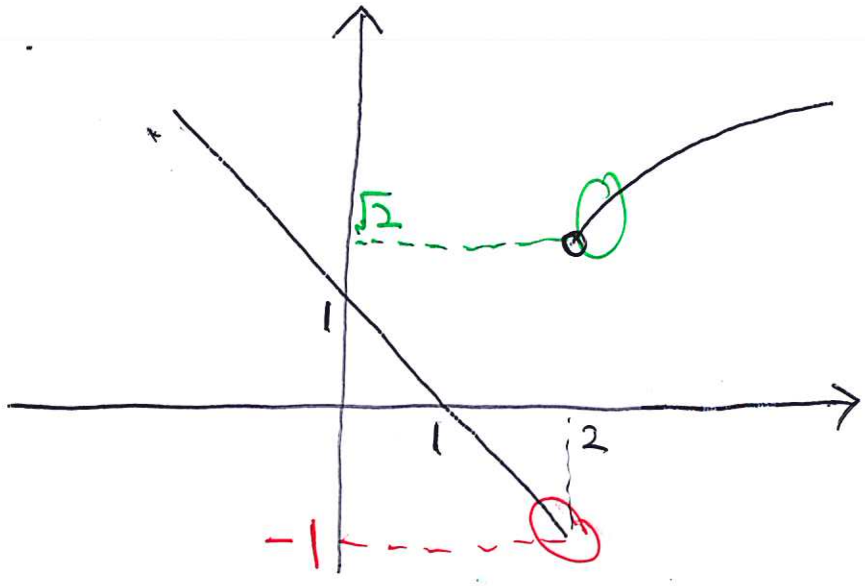
if $x \neq a$ and $|x-a| < \delta$

then $|f(x) - L| < \varepsilon$



eg

$$f(x) = \begin{cases} 1-x & \text{if } x \leq 2 \\ \sqrt{x} & \text{if } x > 2 \end{cases}$$



$\lim\limits_{x \to 2} f(x) = ?$

Sol

$\lim\limits_{x \to 2^-} f(x) = ?$

When $x < 2$, $f(x) = 1 - x$

$\Rightarrow \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} 1 - x = 1 - (2) = \boxed{-1}$

$\lim\limits_{x \to 2^+} f(x) = ?$

When $x > 2$, $f(x) = \sqrt{x}$

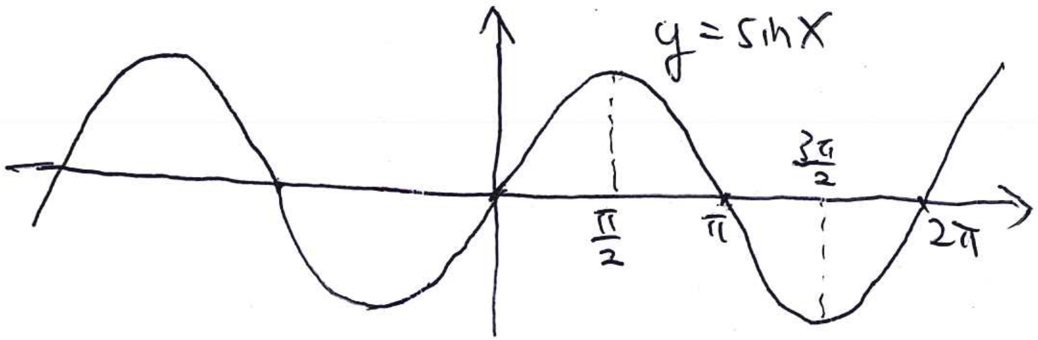
$\Rightarrow \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \sqrt{x} = \boxed{\sqrt{2}}$

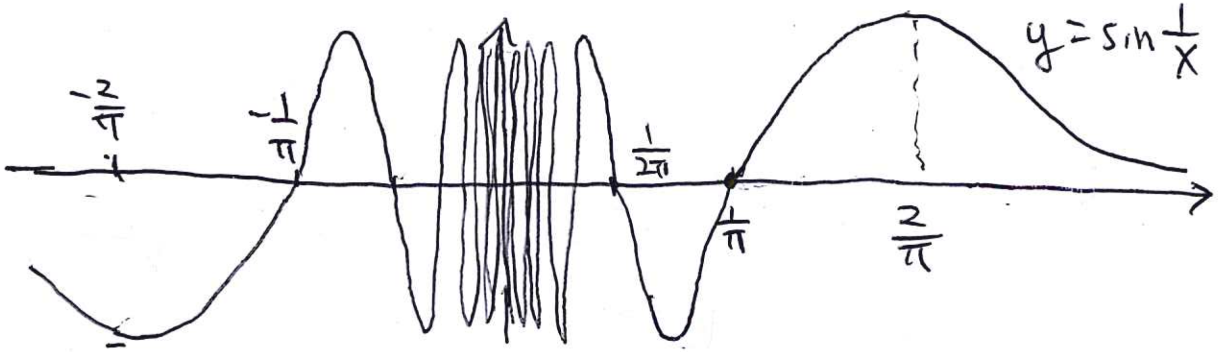
Both $\lim\limits_{x \to 2^-} f(x)$ and $\lim\limits_{x \to 2^+} f(x)$ exist

but they are not equal

$\Rightarrow \lim\limits_{x \to 2} f(x)$ DNE eg $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$

$f(x) = \sin\frac{1}{x}$ $\quad \lim_{x\to 0} f(x) = ?$





When $x \to 0$ from the left, $f(x)$ keeps varying between -1 and 1

$\Rightarrow \lim_{x\to 0^-} f(x)$ DNE

Similarly, $\lim_{x\to 0^+} f(x)$ DNE

So, $\lim_{x\to 0} f(x)$ DNE

Q2 What happens when $x \to \infty$?

Ans As $x \to \infty$,

$\frac{1}{x} \to 0$ from the right

$\sin\frac{1}{x} \to 0$

$\therefore \lim_{x\to\infty} \sin\frac{1}{x} = 0$

Meaning: $f(x)$ is very close to 0 when x is a very large positive number

Similarly, $\lim_{x\to -\infty} \sin\frac{1}{x} = 0$

eg $f(x) = \frac{1}{(x-1)^2}$

$\lim_{x \to 1} \frac{1}{(x-1)^2} = \boxed{\infty}$

very small positive number as $x \to 1$

$\lim_{x \to \infty} \frac{1}{(x-1)^2} = \boxed{0}$

very large positive number as $x \to \infty$, $x \to -\infty$

$\lim_{x \to -\infty} \frac{1}{(x-1)^2} = 0$

Rmk We say that the line $x=1$ is a $\boxed{\text{vertical asymptote}}$ of $f(x)$ and $g(x)$

the line $y=0$ is a $\boxed{\text{horizontal asymptote}}$ of $f(x)$ and $g(x)$

eg $g(x) = \frac{1}{x-1}$

$\lim_{x \to 1^-} \frac{1}{x-1} = \boxed{-\infty}$

very small negative number

$\lim_{x \to 1^+} \frac{1}{x-1} = \infty$

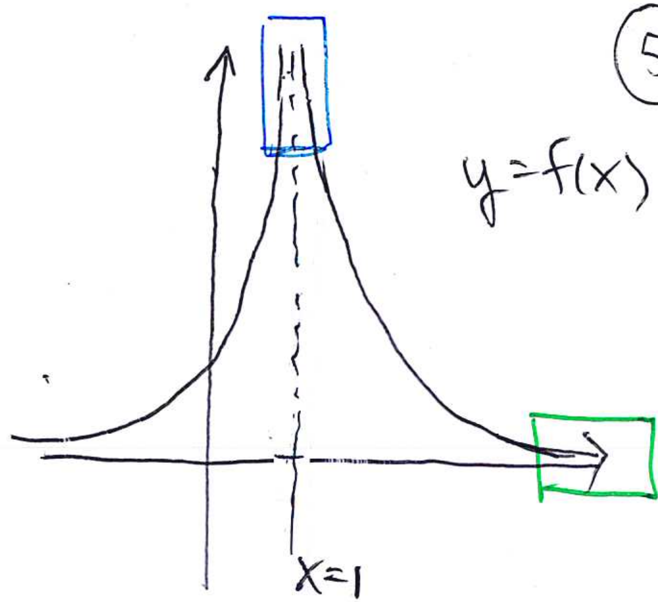
very small positive number

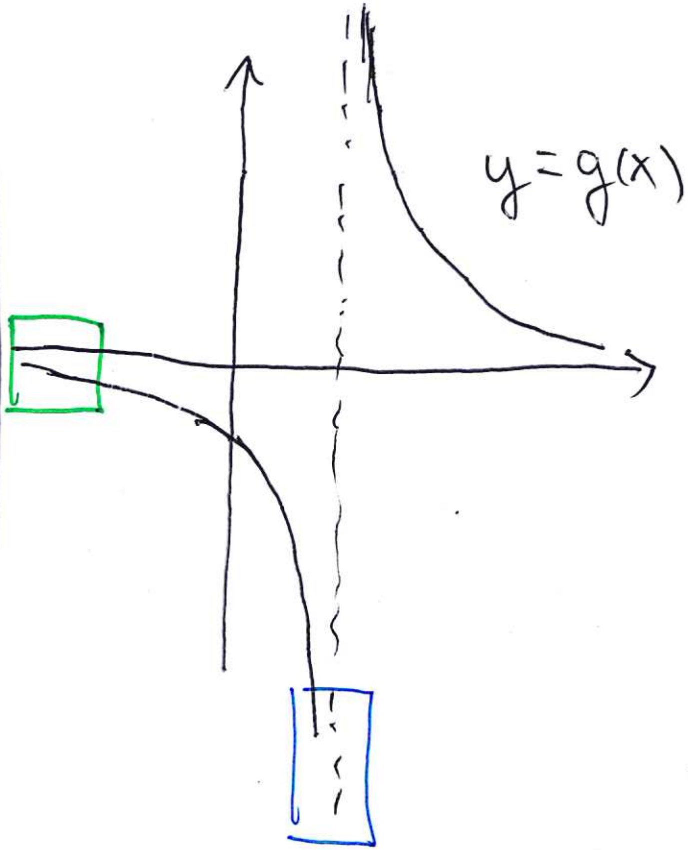
$\lim_{x \to \infty} \frac{1}{x-1} = 0$

very large positive number

$\lim_{x \to -\infty} \frac{1}{x-1} = \boxed{0}$

very large negative number



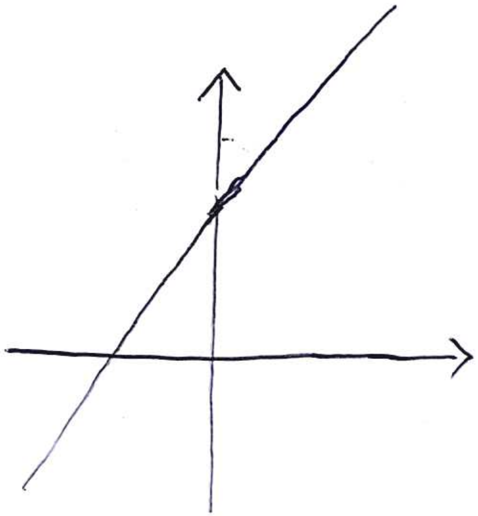


eg $f(x) = \dfrac{x^2+3x-4}{x-1}$

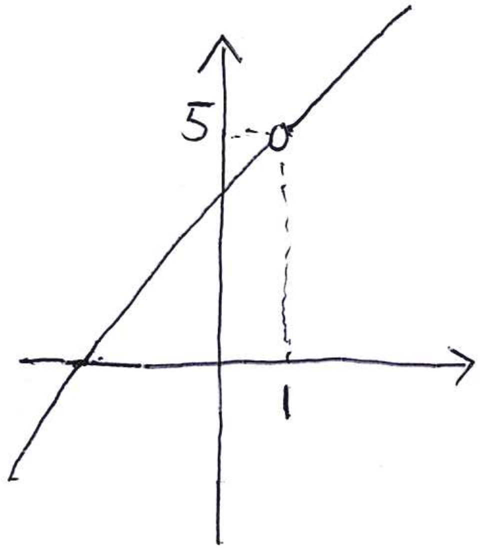
$\lim\limits_{x\to1} f(x) = ?$

$f(x) = \dfrac{(x+4)(x-1)}{x-1}$

$\boxed{\text{if } x \neq 1} = x+4$



$y = x+4$



$y = f(x)$

When $x \neq 1$, $f(x) = x+4$

$\lim\limits_{x\to1} f(x) = \lim\limits_{x\to1} x+4$

$= 1+4$

$= 5$

$+, -, \times, \div$ and power of limit of functions (and also one-sided)

Similar to results to sequence:

eg • $\lim\limits_{x\to a}\big(f(x)-g(x)\big) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$

• $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to\infty} f(x)}{\lim\limits_{x\to\infty} g(x)}$ if $\lim\limits_{x\to\infty} g(x) \neq 0$

• $\lim\limits_{x\to a^+} f(x)^k = \left(\lim\limits_{x\to a^+} f(x)\right)^k$

Compute limits:

eg. $(\pm\infty \pm \infty)$

$$\lim_{x\to 0}\left(\frac{1}{2x}-\frac{1}{x^2+2x}\right)$$

$$=\lim_{x\to 0}\left(\frac{1}{2x}-\frac{1}{x(x+2)}\right)$$

$$=\lim_{x\to 0}\frac{x+2-2}{2x(x+2)}$$

$$=\lim_{x\to 0}\frac{x}{2x(x+2)}$$

$$=\lim_{x\to 0}\frac{1}{2(x+2)}$$

$$=\frac{1}{\lim_{x\to 0}2(x+2)}$$

$$=\frac{1}{2(0+2)}=\frac{1}{4}$$

eg.

$$\lim_{x\to 2}\frac{2-x}{3-\sqrt{x^2+5}}\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 2}\frac{2-x}{3-\sqrt{x^2+5}}\cdot\frac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}}$$

$$=\lim_{x\to 2}\frac{(2-x)(3+\sqrt{x^2+5})}{3^2-(\sqrt{x^2+5})^2}$$

$$=\lim_{x\to 2}\frac{(2-x)(3+\sqrt{x^2+5})}{4-x^2}$$

$$=\lim_{x\to 2}\frac{(2-x)(3+\sqrt{x^2+5})}{(2-x)(2+x)}$$

$$=\lim_{x\to 2}\frac{3+\sqrt{x^2+5}}{2+x}$$

$$=\frac{\lim_{x\to 2}(3+\sqrt{x^2+5})}{\lim_{x\to 2}(2+x)}$$

$$=\frac{3+\sqrt{2^2+5}}{4}$$

$$=\frac{6}{4}$$

$$=\frac{3}{2}$$

e again

limit of a sequence

Last time: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

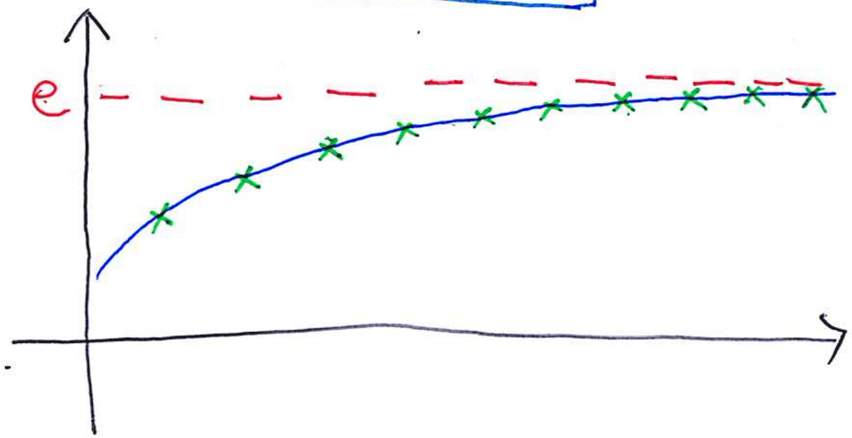
$2, 2.25, 2.37, 2.441, \cdots \to e$

Function version

limit of a function

$$\boxed{e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}}$$

same, $\to \infty$



Rmk $\left(1+\frac{1}{x}\right)^x$ is increasing for $x>0$

eg

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x}$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{2}}\right)^{3x}$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}\cdot 6}$$

$$= \lim_{x\to\infty}\left[\left(1+\frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}}\right]^6$$

$$= \left[\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{2}}\right)^{\frac{x}{2}}\right]^6$$

same, $\to \infty$

$$= e^6$$

eg

$$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x}$$

$$=\lim_{x\to\infty}\left(\frac{x-1+2}{x-1}\right)^{x}$$

$$=\lim_{x\to\infty}\left(1+\frac{2}{x-1}\right)^{x}$$

$$=\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1+1}{2}\cdot 2}$$

$$=\lim_{x\to\infty}\left[\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}+\frac{1}{2}}\right]^{2}$$

$$=\lim_{x\to\infty}\left[\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}}\cdot\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{1}{2}}\right]^{2}$$

$$=\left(\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}}\cdot\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{1}{2}}\right)^{2}$$

$$=\left(e\cdot(1+0)^{\frac{1}{2}}\right)^{2}$$

$$=e^{2}$$

As $x\to\infty$, $\frac{x-1}{2}\to\infty$

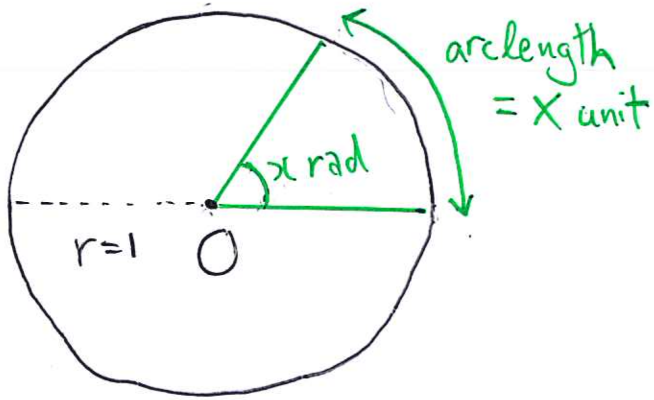
$$a^{m+n}=a^{m}\cdot a^{n}$$

$$(a^{m})^{n}=a^{mn}$$

Radian

- a unit for measuring angles

unit circle (radius = 1)



360° = full circle = 2π rad

180° = half circle = π rad

90° = right angle = $\frac{\pi}{2}$ rad

$\frac{180x}{\pi}^\circ = x$ rad

$y^\circ = \frac{\pi y}{180}$ rad

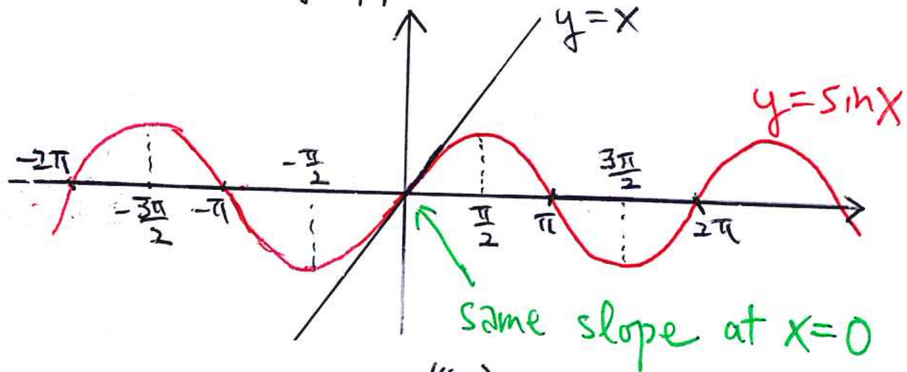
$\boxed{\text{Important: We use radian, not degree, in calculus}}$

Thm $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$

Rmk ① $\sin x = \sin(x \text{ rad})$

② $\lim_{x\to 0} \sin x = 0$, $\lim_{x\to 0} x = 0$

Thm $\Rightarrow$ they approach 0 at similar rate.



eg $\lim_{x\to 0} \frac{\sin(4x)}{\sin(3x)} = \lim_{x\to 0} \frac{\frac{\sin(4x)}{4x}}{\frac{\sin(3x)}{3x}} \cdot \frac{4x}{3x}$

Let $y = 4x$

As $x\to 0$, $y \to 0$

$\lim_{x\to 0} \frac{\sin(4x)}{4x} = \lim_{y\to 0} \frac{\sin y}{y}$

$= 1$

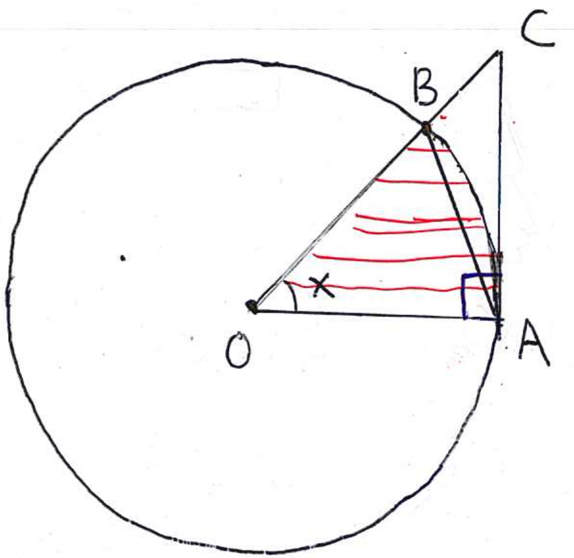
$= \frac{\lim_{x\to 0} \frac{\sin(4x)}{4x}}{\lim_{x\to 0} \frac{\sin(3x)}{3x}} \cdot \frac{4}{3} = \frac{1}{1} \cdot \frac{4}{3}$

$= \frac{4}{3}$

Pf of $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

For $0 < x < \frac{\pi}{2}$

Consider the unit circle

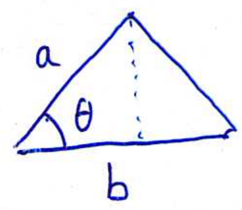


Area of sector OAB

$= \frac{x}{2\pi} \cdot$ Area of whole circle

$= \frac{x}{2\pi} \cdot (1)^2 \pi = \frac{x}{2}$

Area of $\Delta OAB \leq$ Area of sector $OAB \leq$ Area of ΔOAC



Area $= \frac{1}{2} ab \sin \theta$

$$\frac{1}{2} \sin x \leq \frac{x}{2} \leq \frac{1}{2} \tan x$$

$\times \frac{2}{\sin x}$

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

$$\lim_{x \to 0^+} 1 = \lim_{x \to 0^+} \frac{1}{\cos x} = 1$$

Sandwich theorem $\Rightarrow \lim\limits_{x \to 0^+} \frac{x}{\sin x} = 1$

$$\Rightarrow \lim_{x \to 0^+} \frac{\sin x}{x} = \frac{1}{1} = 1 \quad (*)$$

Note that $\frac{\sin x}{x}$ is an even function, i.e. $f(x) = f(-x)$

let $y = -x$. Then $x \to 0^- \Leftrightarrow y \to 0^+$

$$\lim_{x \to 0^-} \frac{\sin x}{x} = \lim_{y \to 0^+} \frac{\sin(-y)}{-y} = \lim_{y \to 0^+} \frac{-\sin y}{-y} = \lim_{y \to 0^+} \frac{\sin y}{y} = 1 \quad (**)$$

$$(*) + (**) \Rightarrow \lim_{x \to 0} \frac{\sin x}{x} = 1$$

eg

$$\lim_{x\to 0} \frac{1-\cos x}{x}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0}(1+\cos x)}$$

$$= (1) \cdot \frac{0}{2} = 0$$